Int. J. Heat Mass Transfer. Vol. 28, No. 2, pp. 506–508, 1985
Printed in Great Britain

0017-9310/85 $3.00 + 0.00

Heat transfer in wavy liquid films

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(*Received* 19 *December* 1983 *and in revised form* 15 *May* 1984)

INTRODUCTION

THE ENERGY equation describing the transfer of heat through a laminar fluid layer with a sinusoidally varying thickness has been solved for Reynolds numbers from 35 to 472 and for Prandtl numbers of 1.7 and about 7. The motivation for this solution was the appraisal of the contribution of convection to the total transfer in such a layer. Hirschburg and Florshuetz [1] have calculated, for approximately sinusoidal and for more complicated wave forms, the heat transfer coefficient for evaporation or condensation based on unidimensional conduction as the only transfer mechanism and have shown a favorable comparison of the prediction to experimental data. These data are of course also predicted fairly well by the empirical specification of Kutadeladze [2] and of Zazuli, as given by Katadeladze [3]. The present question is, however, about the relative contribution of the fluid motion to the transport.

ANALYSIS

The energy equation is

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right). \tag{1}$$

To specify the velocities, the layer thickness is assumed to be

$$\delta=\bar{\delta}(1+\phi) \tag{2}$$

where $\bar{\delta}$ is the average film thickness and ϕ is the local equilibrium wave amplitude. The velocity in the x-direction is assumed to be

$$u=3\bar{u}\left(\eta-\frac{\eta^2}{2}\right) \tag{3}$$

where $\bar{u}$ is the average local velocity and $\eta = y/\delta$. Then the continuity equation gives

$$v=-\int_0^y \frac{\partial u}{\partial x}\,\mathrm{d}y=-3\left[\frac{\partial\bar{u}}{\partial x}\delta\left(\frac{\eta^2}{2}-\frac{\eta^3}{6}\right)-\bar{u}\frac{\partial\delta}{\partial x}\left(\frac{\eta^2}{2}-\frac{\eta^3}{3}\right)\right] \tag{4}$$

also

$$\frac{\partial\delta}{\partial t}=-\frac{\partial\int_0^\delta u\,\mathrm{d}y}{\partial x}=-\frac{\partial\bar{u}\delta}{\partial x} \tag{5}$$

and for a periodic wave with velocity, c

$$\frac{\partial u}{\partial t}=-c\frac{\partial\bar{u}}{\partial x}\quad\text{and}\quad\frac{\partial\delta}{\partial t}=-c\frac{\partial\delta}{\partial x}. \tag{6}$$

Then from equations (2) and (6), and integration of the result

$$(c-\bar{u})(1+\phi)=(c-\bar{u}_0) \tag{7}$$

where $\bar{u}_0$ is the mean velocity for a layer of constant thickness equal to $\bar{\delta}$. Solving for $\bar{u}$ and approximating this result for $\phi < 1$ gives

$$\bar{u}=\bar{u}_0+(c-\bar{u}_0)\phi-(c-\bar{u}_0)\phi^2. \tag{8}$$

Using equation (8) for $\bar{u}$ and equation (2) for δ, gives from equation (4)

$$v=-3\bar{u}_0\bar{\delta}\frac{\partial\phi}{\partial x}\left\{-\left(\frac{c}{\bar{u}_0}-1\right)(1-2\phi)\frac{\delta}{\bar{\delta}}\left(\frac{\eta^2}{2}-\frac{\eta^3}{6}\right)+\left(1+\left(\frac{c}{\bar{u}_0}-1\right)\phi-\left(\frac{c}{\bar{u}_0}-1\right)\phi^2\right)\left(\frac{\eta^2}{2}-\frac{\eta^3}{3}\right)\right\}. \tag{9}$$

With the assumption of a sinusoidal wave, $\phi = A\sin(2\pi/\lambda\,(x-ct))$. The independent variables of the energy equation are transformed from (t, x, y) to (ξ, η) where $\xi = 2\pi/\lambda(x-ct)$, to give

$$C_1\frac{\partial T}{\partial\xi}+C_2\frac{\partial T}{\partial\eta}=C_3\frac{\partial^2 T}{\partial\eta^2}+C_4\frac{\partial^2 T}{\partial\eta\partial\xi}+C_5\frac{\partial^2 T}{\partial\xi^2} \tag{10}$$

where

$$C_1=\frac{2\pi}{\lambda}(u-c)$$

$$C_2=\frac{2\pi}{\lambda}\frac{\bar{\delta}}{\delta}(c-u)A\cos\xi-\left(\frac{2\pi}{\lambda}\right)^2\frac{\bar{\delta}}{\delta}\alpha\eta A\sin\xi-2\left(\frac{2\pi}{\lambda}\right)^2\alpha\left(\frac{\bar{\delta}}{\delta}\right)^2\eta A^2\cos^2\xi+\frac{v}{\delta}$$

$$C_3=\frac{\alpha}{\delta^2}+\alpha\left(\frac{2\pi}{\lambda}\right)^2\left(\frac{\bar{\delta}}{\delta}\right)^2A^2\cos^2\xi$$

$$C_4=-2\alpha\left(\frac{2\pi}{\lambda}\right)^2\eta\frac{\bar{\delta}}{\delta}A\cos\xi$$

$$C_5=\alpha\left(\frac{2\pi}{\lambda}\right)^2.$$

These coefficients are evaluated with u from equations (3) and (8), and v from equation (9).

In the evaporation problem, with a fluid of high latent heat of vaporization, $\bar{\delta}$, the average film thickness over a wave length, will not vary substantially with distance x, and far from the location at which heating or cooling begins, the temporal average temperature profile will be invariable with x. For condensation the average thickness varies more, but the assumption about the temperature profiles is still justifiable. Thus these cases are approximated by the following boundary conditions where the x-axis is along the wall and the y-axis perpendicular to it with $y = 0$ coincides on the wall

$$T(0, \eta) = T(2\pi, \eta)$$
$$T(\xi, 0) = 1$$
$$T(\xi, 1) = 0.$$

Given the solution of equation (10) for these conditions, the heat flux at the wall is

$$q_0 = \frac{-k}{\delta}\frac{\partial T}{\partial \eta}\bigg|_0 \quad \text{and} \quad h_0 = \frac{q_0}{T_0 - T_1} = \frac{q_0}{1}. \tag{11}$$

The heat flux at $\eta = 1$ is the flux normal to the wavy surface, $\mathbf{q} \cdot \mathbf{n}$, and this gives

$$\frac{q}{k} = \left[\frac{2\pi}{\lambda}\left(\frac{\delta}{\delta} A \cos \xi\right)\mathbf{e}_x - \frac{1}{\delta}\mathbf{e}_y\right]\frac{\partial T}{\partial \eta}\bigg|_1 \tag{12}$$

$\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors along the x- and y-axis, respectively. The average heat flow in $0 > \xi > 2\pi$ is obtained by integration. For the wavy layer this is with respect to the actual surface length. For small amplitudes and large wave lengths the average flux for the surface ($\eta = 1$) is approximately

$$\frac{\bar{q}_1}{k} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\delta}\frac{\partial T}{\partial \eta}\bigg|_1 \left[1 + \left(\frac{\partial \delta}{\partial x}\right)^2\right]^{1/2} \mathrm{d}\xi \tag{13}$$

and for the wall it is

$$\frac{\bar{q}_0}{k} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\delta}\frac{\partial T}{\partial \eta}\bigg|_0 \mathrm{d}\xi. \tag{14}$$

RESULTS

The calculations were made, using 40 increments in both ξ and η, for the values of, $2\pi/\lambda$, A, and c/u_0, listed in Table 1, which are experimental determinations made by the cited authors. In Table 1, Columns 1–7 give the experimental conditions and measurements. Column 8 gives the Prandtl numbers for which the calculations are made. One, of the order of 7, corresponds to the experimental conditions, and the other, 1.7, was selected for a comparison to show the effect of Prandtl number. Column 9 gives the calculated average Nusselt number at the wall, and Column 10 gives the average Nusselt number at the outer edge of the layer. These should be the same, and the difference indicates the failure of the calculation to satisfy the energy balance. This difference is small; it increases with Reynolds numbers, reflecting some increase in truncation error. Column 11 gives the average Nusselt number as evaluated for unidimensional conduction.

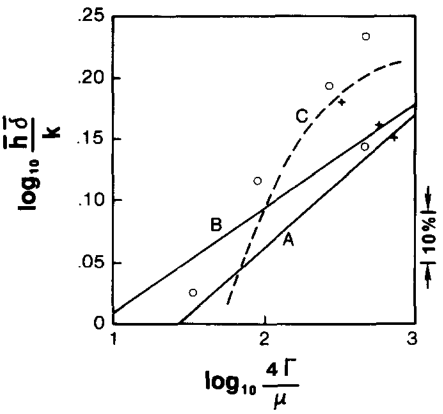

FIG. 1. The average Nusselt number as a function of the Reynold's number Curve A, ref. [2], Curve B, ref. [3], Curve C, ref. [1], for $f^+ = 0.65$.

On this basis, the local Nusselt number is $h\delta/k = 1$ and then

$$\frac{\bar{h}}{k} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\xi}{\delta}. \tag{15}$$

For the sinusoidal wave

$$\frac{\bar{h}}{k} = \frac{1}{2\pi\bar{\delta}}\int_0^{2\pi} \frac{\mathrm{d}\xi}{1 + A \sin \xi} = \frac{1}{2\pi\bar{\delta}}\left[\int_0^{\pi} \frac{\mathrm{d}\xi}{1 + A \sin \xi} + \int_0^{\pi} \frac{\mathrm{d}\xi}{1 - A \sin \xi}\right] = \frac{1}{\bar{\delta}}\frac{1}{\sqrt{(1 - A^2)}}. \tag{16}$$

The difference between the calculated average Nusselt numbers is due to the contribution of convection and to the two-dimensional nature of the conduction that exists because of the variation of layer thickness. It is of interest to investigate the result for $u = v = c = 0$, which corresponds to the two-dimensional conduction solution for the wave. The Nusselt numbers for such a calculation are shown in Columns 12 and 13. These should be equal, and the difference between them indicates the effect of the truncation error in the calculation, made in the same way as for the cases for which there is fluid motion. These values are essentially the same as that of Column 11, for unidimensional conduction, and this correspondence shows that two-dimensional conduction effects are negligible. Then the difference between Nusselt numbers in Columns 9 and 11 reflect only the effect of convection.

Figure 1 shows by circles the values of the Nusselt number,

Table 1

	1	2	3	4	5	6	7	8	9	10	11	12	13
Ref	Temp °C	$\frac{4\Gamma}{\mu}$	$\bar{\delta} \times 10^4$ ft	$\bar{u}_0$ ft/sec	$\frac{2\pi}{\lambda}$ ft^{-1}	$\frac{c}{\bar{u}_0}$	A	$\frac{\nu}{\alpha}$	$\frac{\bar{h}\bar{\delta}}{k}\Big\|_0$	$\frac{\bar{h}\bar{\delta}}{k}\Big\|_1$	$\frac{1}{\sqrt{1-A^2}}$	Conduction $\frac{\bar{h}\bar{\delta}}{k}\Big\|_0$	Conduction $\frac{\bar{h}\bar{\delta}}{k}\Big\|_1$
[5]	20	35	4.132	0.279	191	2.12	0.30	7.2	1.059	1.058	1.05	1.049	1.048
								1.7	1.057	1.056		1.049	1.048
[6]	25	92	5.96	0.383	241	1.93	0.52	6.2	1.309	1.314	1.17	1.171	1.169
								1.7	1.296	1.300		1.171	1.169
[5]	20	268	9.41	0.836	159	1.65	0.55	7.2	1.567	1.611	1.19	1.199	1.197
								1.7	1.543	1.580		1.199	1.197
[5]	20	472	11.5	1.164	136	1.55	0.40	7.2	1.393	1.433	1.09	1.092	1.091
								1.7	1.372	1.405		1.092	1.091
							(0.55)	7.0	1.724	1.816	1.19	1.199	1.197
								1.7	1.694	1.772		1.199	1.197

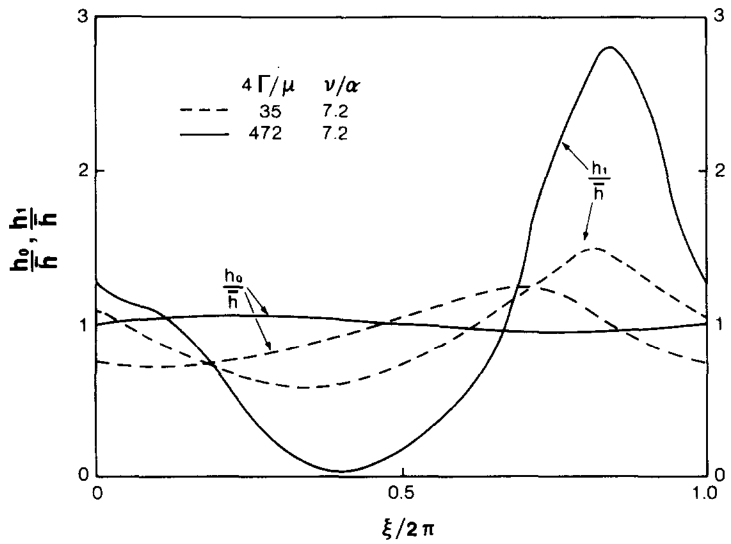

FIG. 2. The local Nusselt number.

$(\bar{h}\delta/k)_0$ as given by Column 9 of Table 1 for a Prandtl number of about 7. The value for $(4\Gamma/\mu)$ of 472 is shown also for an arbitrarily higher value of A, of 0.55, because the value of A probably should not decrease as the Reynolds number increases. This, and also Column 11, shows how important the amplitude is.

Figure 1 also contains lines, A, to show the specification of Zazuli [3] and, B, to show that of Kutadeladze [2]. Curve C is that of Hirschburg and Florschuetz [1] for the intermediate wave solution designated by them as $f^+ = 0.65$ which fits better the average of the data. Three data points from Chun [4] for evaporation of water with a Prandtl number of about 5.6 are shown by plus symbols.

Figure 2 shows the variation of the local heat transfer coefficient, normalized with respect to the average value, as a function of ξ, the results being for the Prandtl number of 7.2 and the Reynolds numbers of 35 and 472. (The same portrayal for the Prandtl number of 1.7 is not much different.) The ratio $h_0/\bar{h}$, for the wall, varies considerably for the low Reynolds number, but not very much for the high Reynolds number. For the surface, the ratio $h_1/\bar{h}$ varies substantially for both cases.

SUMMARY

The present results for the Nusselt numbers are of the order of, but tending to be higher than, the correlation equations of ref. [2, 3]. Comparison of Columns 9 and 11 indicates a considerable effect of convection and two-dimensional conduction.

The present results are of the order of those given by ref. [1] for an intermediate wave, but an account of convection and two-dimensional conduction in those results would probably require a value of f^+ closer to unity (approaching a sinusoidal wave form) to maintain correspondence with the data.

The present results for a Prandtl number of 1.7 indicate little effect of the Prandtl number. This indication is in accord with the inferences of the correlation of refs. [2,3], and of ref. [1], since the Nusselt number obtained from equation (15) is independent of the Prandtl number for a given fluid viscosity.

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